2.5 HOMOGENEOUS 1ST ORDER ODE

DEF'N: $f(x,y)$ IS HOMOGENEOUS OF ORDER n

IF $f(tx,ty) = t^n f(x,y)$

DEF'N: $M\,dx + N\,dy = 0$ IS HOMOGENEOUS

IF BOTH M, N ARE HOMOGENEOUS OF THE SAME ORDER

$$\underbrace{\underbrace{(y^3 - xy^2)}_{M}\,dx + \underbrace{2x^2y}_{N}\,dy = 0}_{\text{HOMOGENEOUS}}$$

BOTH HOMOGENEOUS OF ORDER 3

$$\begin{aligned} M(tx,ty) &= (ty)^3 - (tx)(ty)^2 \\ &= t^3y^3 - t^3xy^2 \\ &= t^3(y^3 - xy^2) = t^3M(x,y) \end{aligned}$$

$$\begin{aligned} N(tx,ty) &= 2(tx)^2(ty) \\ &= 2t^3x^2y \\ &= t^3(2x^2y) = t^3N(x,y) \end{aligned}$$

IF $M\,dx + N\,dy = 0$ IS HOMOGENEOUS,

LETTING $y = v(x)x$ (OR $x = v(y)y$)

WILL TRANSFORM THE DE INTO A SEPARABLE DE

$$\begin{aligned} y &= vx \\ dy &= dv \cdot x + v \cdot dx \\ dy &= x\,dv + v\,dx \end{aligned}$$

$$\begin{aligned} x &= vy \\ &\vdots \\ dx &= y\,dv + v\,dy \end{aligned}$$

$$(y^3 - xy^2)\,dx + 2x^2y\,dy = 0$$

LET $\boxed{y = vx}$

$dy = x\,dv + v\,dx$

$$\boxed{((vx)^3 - x(vx)^2)\,dx + 2x^2(vx)\boxed{x\,dv + v\,dx}} = 0$$

$$(v^3x^3 - \underline{v^2x^3 + 2v^2x^3})\,dx + (2vx^4 \qquad)\,dv = 0$$

$$(v^3x^3 + v^2x^3)\,dx + 2vx^4\,dv = 0$$

$$\cancel{vx^3}((v^2+v)\,dx + 2x\,dv \quad) = 0$$

COULD $v = 0$?

$y = 0x = 0$

IS A SOL'N OF DE

$$\boxed{(v^2+v)\,dx + 2x\,dv = 0}$$

$$2x\,dv = -(v^2+v)\,dx$$

$$\boxed{\int \frac{2}{v^2+v}\,dv = \int -\frac{1}{x}\,dx} \longleftarrow$$

MANDATORY CHECKPOINT: SEPARABLE

$$\frac{2}{v^2+v} = \frac{A}{v} + \frac{B}{v+1}$$

$$2 = A(v+1) + Bv$$

$v = 0$: $2 = A(1) \rightarrow A = 2$

$v = -1$: $2 = -B \rightarrow B = -2$

SANITY CHECK: $v = 2$: $\frac{2}{4+2} \overset{?}{=} \frac{2}{2} + \frac{-2}{3}$

$\frac{1}{3} = \frac{2}{6} \overset{?}{=} 1 - \frac{2}{3} = \frac{1}{3}$ ✓

$$\int\left(\frac{2}{v} - \frac{2}{v+1}\right)dv = -\ln|x| + C$$

$$\boxed{2\ln|v| - 2\ln|v+1| = -\ln|x| + C}$$

$e \qquad e$

$$\boxed{\left(\frac{v}{v+1}\right)^2 = \frac{C}{x}}$$

$$\left(\frac{v}{v+1}\right)^2 = \frac{C}{x}$$

$$\left(\frac{v+1}{v}\right)^2 = Cx$$

$$\frac{v+1}{v} = C\sqrt{x}$$

$$1 + \frac{1}{v} = C\sqrt{x}$$

$$v = \frac{1}{C\sqrt{x} - 1}$$

$$\boxed{y = vx = \frac{x}{C\sqrt{x} - 1}} \quad \text{OR } y = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad M(tx,ty) = t^n M(x,y)$$

$$N(tx,ty) = t^n N(x,y)$$

LET $y = vx$

$dy = x\,dv + v\,dx$

$$M(x,vx)\,dx + N(x,vx)(x\,dv + v\,dx) = 0$$

$$\cancel{x^n} M(v)\,dx + \cancel{x^n} N(v)(x\,dv + v\,dx) = 0$$

$$M(x,vx) = M(x\cdot 1, x\cdot v) = x^n M(1,v) = x^n M(v)$$

$$(M(v) + vN(v))\,dx + x\,N(v)\,dv = 0$$

$$x\,N(v)\,dv = -(M(v) + vN(v))\,dx$$

$$\int \frac{N(v)}{M(v)+vN(v)}\,dv = \int -\frac{1}{x}\,dx$$

SEPARABLE

DON'T USE THIS FORMULA AS A SHORTCUT

$$\frac{dy}{dx} = \frac{y^2-x^2}{xy}$$

$$xy\,dy = (y^2-x^2)\,dx$$

$$(\underbrace{x^2-y^2}_{M})\,dx + \underbrace{xy}_{N}\,dy = 0$$

$$M(tx,ty) = (tx)^2-(ty)^2 = t^2(x^2-y^2) = t^2 M(x,y)$$

$$N(tx,ty) = (tx)(ty) = t^2(xy) = t^2 N(x,y)$$

LET $\cancel{x=vy}$ $y=vx$

$$dx = y\,dv + v\,dy$$

$$\cancel{(v^2y^2-y^2)\,dx + vy^2\,dy = 0}$$

$$\cancel{(v^2-1)\,dx + v\,dy = 0}$$

$$(v^2y^2-y^2)(y\,dv+v\,dy) + vy^2\,dy = 0$$

$$(v^2-1)(y\,dv+v\,dy) + v\cancel{y^2}\,dy = 0$$

$$(v^2-1)y\,dv + (v^3-v+v)\,dy = 0$$

$$(v^2-1)y\,dv + v^3\,dy = 0$$

$$(v^2-1)y\,dv = -v^3\,dy$$

$$\left(\frac{v^2-1}{v^3}\right)dv = -\frac{1}{y}\,dy$$

$$\int\left(\frac{1}{v}-\frac{1}{v^3}\right)dv = -\int\frac{1}{y}\,dy$$

$$\ln|v| + \frac{1}{2}v^{-2} = -\ln|y| + C$$

$v = ?$